The theoretical and numerical solutions coincided to an accuracy of 10^{-4} at step values of $h_1 = 0.09$, $h_2 = 0.1$. Twelve points in time with a step $\Delta t = 0.1$ were calculated. Time required for calculating one variant was 5 min.

NOTATION

γ and $\varkappa$, coefficients; u, temperature, concentration, etc.; x and y, coordinates; t, time; Δ, Laplace operator; f(x, y), initial function; g(x, y, t) and $T_1(y, t)$; $T_2(x, t)$; $T_3(y, t)$; $T_4(x, t)$; $\varphi_1(y, t)$; $\varphi_2(x, t)$; $\varphi_3(y, t)$; $\varphi_4(x, t)$, specified functions; s_1 and s_2, dimensionality coefficients of data blocks; $A^{(0)}$, $A^{(t)}$, $A^{(t-1)}$, matrices of algebraic system of equations for boundary conditions of first sort; $B^{(0)}$, $B^{(t)}$, B^{t-1}, matrices of system of equations for boundary conditions of second sort; g_i, vectors of right-hand sides; E, unit matrix; α, β, elements of three-diagonal matrix; μ_j, $\mu_{j-2}{}^{t+1}$, $u_{j+2}{}^{t+1}$, vectors of desired quantities; $q_i{}^{(t)}$, $p_i{}^{(t)}$, $t = 1, \ldots, s$ vectors in cyclical reduction; h_1 and h_2, step of space grid; Δt, step in time; a, b, dimensions of rectangle; N_1 and N_2, number of grid points along x and y axes, respectively; z^n_{ij}, grid; u^n_{ij}, grid function; σ, real parameter (weight).

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SOLUTION OF THE NONLINEAR INVERSE THERMAL CONDUCTIVITY PROBLEM BY THE ITERATION METHOD

O. M. Alifanov and V. V. Mikhailov

A regular iteration algorithm is constructed for the case of a nonlinear generalized thermal conductivity equation for determination of the nonstationary thermal flux. The algorithm is based on the method of conjugate gradients.

In experimental studies of nonstationary thermal processes, it becomes necessary to calculate thermal boundary conditions from temperature measurements within bodies (the inverse thermal conductivity boundary problem). The well-known incorrectness of the formulation of this inverse problem, which manifests itself as a strong sensitivity of the results to errors in the input information, requires the development of approximate algorithms which can suppress the instability of the results and maintain required accuracy.

We will consider the inverse problem for a nonlinear generalized thermal conductivity equation in the region $\{0 \le x \le b,\ 0 \le t \le t_m\}$. It is required that the dependence of thermal flux $q_1(t)$ on the left-hand boundary on the known temperature f(t) and the thermal flux $q_2(t)$ on the right-hand boundary be determined. Initial conditions are specified. Thus, we have

$$C(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right) + K(T)\frac{\partial T}{\partial x} + \varphi(T),\ 0 < x < b,\ 0 < t \leqslant t_m, \tag{1}$$

$$T(x, 0) = \xi(x), \tag{2}$$

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$$-\lambda(T(0,\ t))\frac{\partial T(0,\ t)}{\partial x}=q_1(t)-?, \tag{3}$$

$$-\lambda(T(b,\ t)\frac{\partial T(b,\ t)}{\partial x}=q_2(t), \tag{4}$$

$$T(b,\ t)=f(t), \tag{5}$$

where $C(T)$, $\lambda(T)$, $K(T)$, $\varphi(T)$, $\xi(x)$, $q_2(t)$, $f(t)$ are known functions.

We now represent the incorrectly formulated problem of Eqs. (1)-(5) as an optimum central problem, i.e., we must choose a $q_1(t)$ for conditions (1)-(4) so as to minimize the function

$$J(q_1(t))=\int_0^{t_m}[T(b,\ t)-f(t)]^2dt. \tag{6}$$

We will seek the solution of this extremal problem by the conjugate gradient method. This method has a superlinear convergence rate and is quite economical in terms of computation time. Compared to the method of most rapid descent or the method of possible first-order directions, the conjugate gradient method is significantly less dependent on the specifications of the extremal problem, i.e., on the form of the region in which the solution is sought [1]. It should also be noted that this method effectively permits commencement of the search for a solution from a far-removed initial approximation, and, while gradually reducing the rate of convergence, makes it possible to "accurately" approach the required approximate solution [2].

We will now obtain a formula for calculation of the gradient of function (6). We assume that the thermal flux $q_1(t)$ varies by a small amount $u(t)$. Then temperature $T(x,\ t)$ varies by an amount $v(x,\ t)$, which satisfies the following

$$\frac{\partial v}{\partial t}=A(x,\ t)\ \frac{\partial^2 v}{\partial x^2}+B(x,\ t)\ \frac{\partial v}{\partial x}+D(x,\ t)v,\ 0<x<b,\ 0<t\leqslant t_m, \tag{7}$$

$$v(x,\ 0)=0, \tag{8}$$

$$-\frac{\partial\lambda(0,\ t)}{\partial x}v(0,\ t)-\lambda(0,\ t)\ \frac{\partial v(0,\ t)}{\partial x}=u(t), \tag{9}$$

$$-\frac{\partial\lambda(b,\ t)}{\partial x}v(b,\ t)-\lambda(b,\ t)\frac{\partial v(b,\ t)}{\partial x}=0, \tag{10}$$

where

$$A(x,\ t)=\lambda(x,\ t)/C(x,\ t);$$

$$B(x,\ t)=\left(\frac{2\partial\lambda(x,\ t)}{\partial x}+K(x,\ t)\right)/C(x,\ t);$$

$$D(x,\ t)=\left(\frac{\partial^2\lambda(x,\ t)}{\partial x^2}+\frac{\partial K(x,\ t)}{\partial x}+\frac{\partial\varphi(x,\ t)}{\partial T}-\frac{\partial C(x,\ t)}{\partial t}\right)/C(x,\ t).$$

The functions $A(x,\ t)$, $B(x,\ t)$, $D(x,\ t)$, $\lambda(0,\ t)$, $\lambda(b,\ t)$ are defined by the solution of Eqs. (1)-(4). Below, to reduce the complexity of notation, we will retain the arguments of the functions only where needed for clarity. For the linear component of the function increment, we have

$$\Delta J=\int_0^{t_m}2\,[T(b,\ t)-f(t)]\,v(b,\ t)\,dt. \tag{11}$$

In order that function (11) take on an extremal value, it is necessary that

$$\Delta I=0, \tag{12}$$

where

$$I=I_0+I_1+I_2,$$

$$I_0 = \Delta J = 2\int_0^{t_m} [T(b, t) - f(t)]\, v(b, t)\, dt,$$

$$I_1 = \int_0^b \eta(x, 0)\, v(x, 0)\, dx$$

$$+ \int_0^{t_m} \eta(0, t)\left[u(t) + \frac{\partial\lambda(0, t)}{\partial x} v(0, t) + \lambda(0, t)\frac{\partial v(0, t)}{\partial x}\right] dt$$

$$+ \int_0^{t_m} \eta(b, t)\left[\frac{\partial\lambda(b, t)}{\partial x} v(b, t) + \lambda(b, t)\frac{\partial v(b, t)}{\partial x}\right] dt,$$

$$I_2 = \int_0^{t_m}\int_0^b \psi(x, t)\left[A\frac{\partial^2 v}{\partial x^2} + B\frac{\partial v}{\partial x} + Dv - \frac{\partial v}{\partial t}\right] dx dt,$$

and $\eta(x, 0)$, $\eta(0, t)$, $\eta(b, t)$, $\psi(x, t)$ are undefined Lagrange factors; ΔI is the complete variation of the function I.

We will consider the individual terms of the left side of Eq. (12):

$$\Delta I_0 = 2\int_0^{t_m} [T(b, t) - f(t)]\, \delta v(b, t)\, dt,$$

$$\Delta I_1 = \int_0^b \eta(x, 0)\, \delta v(x, 0)\, dx + \int_0^{t_m} \eta(0, t)\left[\delta u + \frac{\partial\lambda}{\partial x}\delta v\right.$$

$$\left. + \lambda\delta\left(\frac{\partial v}{\partial x}\right)\right] dt + \int_0^{t_m} \eta(b, t)\left[\frac{\partial\lambda}{\partial x}\delta v + \lambda\delta\left(\frac{\partial v}{\partial x}\right)\right] dt,$$

$$\Delta I_2 = \int_0^{t_m}\int_0^b \left[\frac{\partial^2}{\partial x^2}(A\psi) - \frac{\partial}{\partial x}(B\psi) + D\psi + \frac{\partial\psi}{\partial t}\right]\delta v(x, t)\, dx dt$$

$$+ \int_0^{t_m}\left[A\psi\delta\left(\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial x}(A\psi)\delta v + B\psi\delta v\right]_{x=0}^{x=b} dt - \int_0^b (\psi\delta v)_{t=0}^{t=t_m} dx.$$

Substituting ΔI_0, ΔI_1, and ΔI_2 in Eq. (12), we obtain an expression containing terms dependent on the double integral and integrals over the boundaries of the region under consideration. Considerations usually employed in variation calculation allow us to establish that for the stationary condition (12) to be fulfilled, it is necessary to set equal to zero each of the groups of terms upon variation [3]. Thus, omitting intermediate calculations, we can write the condition of the boundary problem conjugate to that of Eqs. (7)-(10).

$$-\frac{\partial\psi}{\partial t} = \frac{\partial^2}{\partial x^2}(A\psi) - \frac{\partial}{\partial x}(B\psi) + D\psi,\ 0 < x < b,\ 0 < t \leqslant t_m, \tag{13}$$

$$\psi(x, t_m) = 0, \tag{14}$$

$$\frac{\partial\lambda(0, t)}{\partial x}\frac{A(0, t)\psi(0, t)}{\lambda(0, t)} + \frac{\partial}{\partial x}(A(0, t)\psi(0, t)) - B(0, t)\psi(0, t) = 0, \tag{15}$$

$$\frac{\partial\lambda(b, t)}{\partial x}\frac{A(b, t)\psi(b, t)}{\lambda(b, t)} + \frac{\partial}{\partial x}(A(b, t)\psi(b, t)) - B(b, t)\psi(b, t) = 2[T(b, t) - f(t)]. \tag{16}$$

Then for the increment of Eq. (11) with consideration of Eqs. (15), (16) we obtain

$$\Delta J = 2\int_0^{t_m} [T(b, t) - f(t)]\, v(b, t)\, dt = \int_0^{t_m}\left[v\left(\frac{\partial\lambda}{\partial x}\frac{A\psi}{\lambda} + \frac{\partial}{\partial x}(A\psi) - B\psi\right)\right]_{x=0}^{x=b} dt = \int_0^b S_1(x)\, dx + \int_0^b S_2(x)\, dx, \tag{17}$$

where

$$S_1(x) = \int_0^{t_m} v \frac{\partial}{\partial x}\left(\frac{\partial \lambda}{\partial x} \frac{A\psi}{\lambda} + \frac{\partial}{\partial x}(A\psi) - B\psi\right) dt,$$

$$S_2(x) = \int_0^{t_m} \frac{\partial v}{\partial x}\left(\frac{\partial \lambda}{\partial x} \frac{A\psi}{\lambda} + \frac{\partial}{\partial x}(A\psi) - B\psi\right) dt.$$

Further, using the equation for the conjugate variable (13) and integrating by parts with consideration of Eqs. (8), (14), (7), we have

$$\int_0^b S_1(x)\,dx = \int_0^b dx \int_0^{t_m} \psi\left(A\frac{\partial^2 v}{\partial x^2} + B\frac{\partial v}{\partial x}\right) dt + \int_0^b dx \int_0^{t_m} v \frac{\partial}{\partial x}\left(\frac{\partial \lambda}{\partial x} \frac{A\psi}{\lambda}\right) dt.$$

Integrating by parts with use of Eqs. (9), (10) gives

$$\int_0^b S_1(x)\,dx = \int_0^{t_m} \frac{A(0,\ t)\psi(0,\ t)}{\lambda(0,\ t)} u(t)\,dt - \int_0^b S_2(x)\,dx. \tag{18}$$

From Eqs. (17) and (18) it follows that

$$\Delta J = \int_0^{t_m} \frac{A(0,\ t)\psi(0,\ t)}{\lambda(0,\ t)} u(t)\,dt. \tag{19}$$

Now Eq. (19) is the differential [4] of Eq. (6), the gradient of which can be written in the form

$$J'(q_1) = A(0,\ t)\psi(0,\ t)/\lambda(0,\ t). \tag{20}$$

However, as was noted in [5], with use of this formula to calculate the gradient uniform convergence is absent – the gradient $J'(q_1)$ deviates from exactness because of condition (14) in some neighborhood of the point $t = t_m$. The accuracy of establishing the thermal flux in this region is determined to a significant degree by the choice of the initial approximation for $q_1(t)$.

In order to reduce the effect of condition (14) on convergence, we will seek a solution of the equation $q_1(t)$ consisting of a continuously differentiable function corresponding to the condition

$$q_1(t) = \int_0^t \frac{dq_1}{d\tau} d\tau. \tag{21}$$

With consideration of Eq. (14), we can write

$$\frac{d}{dt}\int_{t_m}^t \frac{A(0,\ \tau)\psi(0,\ \tau)}{\lambda(0,\ \tau)} d\tau = \frac{A(0,\ t)\psi(0,\ t)}{\lambda(0,\ t)} = J'(q_1(t)),$$

and then Eq. (19) can be represented in the form

$$\Delta J = \int_0^{t_m} \frac{d}{dt}\left(\int_{t_m}^t \frac{A(0,\ \tau)\psi(0,\ \tau)}{\lambda(0,\ \tau)} d\tau\right) u(t)\,dt = \int_0^{t_m} \frac{du}{dt} \int_t^{t_m} \frac{A(0,\ \tau)\psi(0,\ \tau)}{\lambda(0,\ \tau)} d\tau dt.$$

Thus, we obtain a new formula for the gradient of Eq. (6)

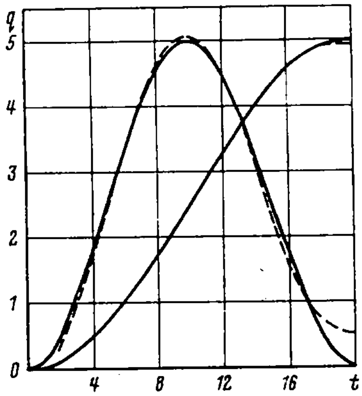

Fig. 1

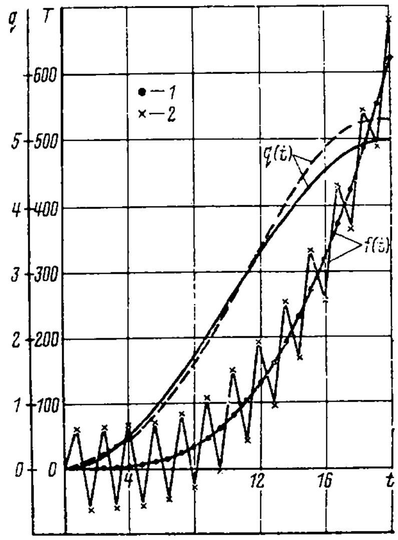

Fig. 2

Fig. 1. Establishment of thermal fluxes q, $W/m^2 \cdot 10^{-5}$ for exact input data: solid line, exact solution; dashed line, approximation (25 iterations). t, time, sec.

Fig. 2. Establishment of thermal flux q, $W/m^2 \cdot 10^{-5}$ for sawtooth perturbation of input data f(t), deg: 1) exact values of input data; 2) perturbed ($\Delta f = \pm 0.1 f_{max}$, j = 1, 2, ..., m); solid curve, solution corresponding to exact input data; dashed line, approximate solution with perturbed input data (eighth iteration).

$$J'\left(\frac{dq_1}{dt}\right) = \int\limits_t^t \frac{A(0,\ \tau)\psi(0,\ \tau)}{\lambda(0,\ t)}\,d\tau. \tag{22}$$

To calculate $J'(dq_1/dt)$ it is necessary to solve successively boundary problems (1)-(4) and (13)-(16).

Iteration approximations for the unknown function are constructed by the method of conjugate gradients with the following formulas:

$$\begin{gathered} q_1^{k+1} = q_1^k - \alpha r^k, \quad k = 0,\ 1,\ 2,\ \ldots, \\ r^k = \int\limits_0^t p^k(\tau)\,d\tau, \quad p^k(\tau) = -J_k' + \beta_k p^{k-1}(\tau), \\ \beta_k = \int\limits_0^{t_m} (J_k' - J_{k-1}',\ J_k')\,dt \Big/ \int\limits_0^{t_m} (J_{k-1}',\ J_{k-1}')\,dt,\ \beta_0 = 0, \end{gathered} \tag{23}$$

where J'_k is calculated with Eq. (22); $q_1^0(t)$ is the known initial approximation.

The coefficients α_k defining the value of the step in the transition from q_1^k to q_1^{k+1} are found from the condition $\min\limits_{\alpha} J(q_1^k - \alpha r^k)$. It should be noted that the method for solution proposed here assumes a known thermal flux value at the initial moment of time $q_1(0) = q_1^0(t)$. We will now turn to the major question of a choice of some reasonable approximation, i.e., we will consider the conditions for completion of iteration process (23). Calculation experiments have revealed that upon freeing of the input data f(t) and $q_2(t)$ from fluctuation errors (smoothing the experimental information) sequence (23) does not give diverging sequences, and to shorten the search process one of the traditional check conditions may be used, e.g.:

$$J(q_1^{k+1}) \leqslant \varepsilon_1, \quad \int\limits_0^{t_m} (J'q_1'^{k+1})^2\,dt \leqslant \varepsilon_2, \quad \max\limits_t |T^{k+1}(b,\ t) - f(t)| \leqslant \varepsilon_3,$$

where ε_1, ε_2, ε_3 are small positive numbers.

However, if function f(t) contains a sufficiently marked oscillating component, then with increasing approximation to the solution corresponding to perturbed input data, the results will lose a regular character more and more. This is the natural behavior of an iteration solution of the incorrectly formulated inverse problem.

Following [2, 5] we will limit the iteration sequence (23) to some number $k = k^*$, at which the inequality $J(q_1^{k^*+1}) < \varepsilon^2$ is first fulfilled, and choose the length of the step in the direction r^{k^*} from the condition

$$\min_{\alpha} \{J(q_1^{k^*} - \alpha r^{k^*}) - \varepsilon^2\},$$

where ε^2 is the integral accuracy of specifying the experimental information ($\varepsilon^2 = \int_0^{t_m} \sigma^2(t)dt$; σ is the mean square error of the temperature measurements).

The proposed iteration method together with choice of the unknown function by the condition of matching the nonbinding to the accuracy of the input data is regular, and guarantees stable results. A theoretical justification of this approach and its relationship to the regularization method [6] and the nonbinding principle [7] may be found in [8].

A program was written on the basis of the above method and calculations performed for a number of methodical examples. The boundary problems (1)-(4), (7)-(10), and (13)-(16) were solved numerically by an implicit difference method. Figures 1 and 2 show results for several examples with the following initial data:

$$A(T) = \lambda(T)/C(T) = 0.4 \cdot 10^{-6} - 0.143 \cdot 10^{-9} \cdot T + 0.408 \cdot 10^{-12} \cdot T^2, \ \mathrm{m^2/sec},$$
$$\lambda(T) = 0.721 + 0.288 \cdot 10^{-3} \cdot T + 0.15 \cdot 10^{-6} \cdot T^2, \ \mathrm{W/m/deg},$$
$$K(T) = \varphi(T) = 0, \qquad b = 0.003 \ \mathrm{m}, \qquad t_m = 20 \ \mathrm{sec}.$$

The initial temperature distribution was taken constant and equal to zero, with the internal boundary ($x = b$) thermally insulating ($q_2(t) = 0$). For the exact and "experimental" information we used the temperature on the inner boundary, obtained by numerical solution of the direct problem for a specified law of change of $q_1(t)$. After perturbation of the inner temperature by some rule the inverse problem was solved. Practical analysis of the correctness allows us to conclude that the method is stable with respect to perturbation of the "input" data, and that the results are independent of the initial approximation.

NOTATION

$C(T)$, volume heat capacity; $\lambda(T)$, thermal conductivity coefficient; $K(T)$, filtration coefficient; $\varphi(T)$, distributed heat source (drain); T, temperature; x, coordinate; t, τ, temperature; $\xi(x)$, initial temperature distribution; q, thermal flux; b, right-hand boundary along x; t_m, right-hand boundary of time interval; $f(t)$, input data; $J(q)$, minimized criterion; v, temperature increment; u, change in thermal flux; I, expanded function; k, number of iteration; β, α, parameters of conjugate gradient method; ε, input temperature error.

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